

DETERMINANT OF PSEUDO-LAPLACIANS

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Abstract. We derive comparison formulas relating the zeta-regularized determinant of an arbitrary self-adjoint extension of the Laplace operator with domain $C_c^\infty(X \setminus \{P\}) \subset L_2(X)$ to the zeta-regularized determinant of the Laplace operator on X . Here X is a compact Riemannian manifold of dimension 2 or 3; $P \in X$.

1. INTRODUCTION

Let X_d be a complete Riemannian manifold of dimension $d \geq 2$ and let Δ be the (positive) Laplace operator on X_d . Choose a point $P \in X_d$ and consider Δ as an unbounded symmetric operator in the space $L_2(X_d)$ with domain $C_c^\infty(X_d \setminus \{P\})$. It is well-known that thus obtained operator is essentially self-adjoint if and only if $d \geq 4$. In case $d = 2, 3$ it has deficiency indices $(1, 1)$ and there exists a one-parameter family $\Delta_{\alpha, P}$ of its self-adjoint extensions (called pseudo-laplacians; see [3]). One of these extensions (the Friedrichs extension $\Delta_{0, P}$) coincides with the self-adjoint operator Δ on X_d . In case $X_d = R^d$, $d = 2, 3$ the scattering theory for the pair $(\Delta_{\alpha, P}, \Delta)$ was extensively studied in the literature (see e. g., [1]). The spectral theory of the operator $\Delta_{\alpha, P}$ on compact manifolds X_d ($d = 2, 3$) was studied in [3], notice also a recent paper [15] devoted to the case, where X_d is a compact Riemann surface equipped with Poincaré metric.

The zeta-regularized determinant of Laplacian on a compact Riemannian manifold was introduced in [11] and since then was studied and used in an immense number of papers in string theory and geometric analysis, for our future purposes we mention here the memoir [5], where the determinant of Laplacian is studied as a functional on the space of smooth Riemannian metrics on a compact two-dimensional manifold, and the papers [6] and [13], where the reader may find explicit calculation of the determinant of Laplacian for three-dimensional flat tori and for the sphere S^3 (respectively).

The main result of the present paper is a comparison formula relating $\det(\Delta_{\alpha, P} - \lambda)$ to $\det(\Delta - \lambda)$, for $\lambda \in \mathbb{C} \setminus (\text{Spectrum}(\Delta) \cup \text{Spectrum}(\Delta_{\alpha, P}))$.

It should be mentioned that in case of two-dimensional manifold the zeta-regularization of $\det(\Delta_{\alpha, P} - \lambda)$ is not that standard, since the corresponding operator zeta-function has logarithmic singularity at 0.

It should be also mentioned that in the case when the manifold X_d is flat in a vicinity of the point P we deal with a very special case of the situation

(Laplacian on a manifold with conical singularity) considered in [10], [8], [9] and, via other method, in [7]. The general scheme of the present work is close to that of [7], although some calculations from [9] also appear very useful for us.

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2. PSEUDO-LAPLACIANS, KREIN FORMULA AND SCATTERING COEFFICIENT

Let X_d be a compact manifold of dimension $d = 2$ or $d = 3$; $P \in X_d$ and $\alpha \in [0, \pi)$. Following Colin de Verdière [3], introduce the set

$$\mathcal{D}(\Delta_{\alpha,P}) = \{f \in H^2(X_d \setminus \{P\}) : \exists c \in \mathbb{C} : \text{in a vicinity of } P \text{ one has}$$

$$(1) \quad f(x) = c(\sin \alpha \cdot G_d(r) + \cos \alpha) + o(1) \text{ as } r \rightarrow 0\},$$

where

$$H^2(X_d \setminus \{P\}) = \{f \in L_2(X_d) : \exists C \in \mathbb{C} : \Delta f - C\delta_P \in L_2(X_d)\},$$

r is the geodesic distance between x and P and

$$G_d(r) = \begin{cases} \frac{1}{2\pi} \log r, & d = 2 \\ -\frac{1}{4\pi r}, & d = 3. \end{cases}$$

Then (see [3]) the self-adjoint extensions of symmetric operator Δ with domain $C_c^\infty(X_d \setminus \{P\})$ are the operators $\Delta_{\alpha,P}$ with domains $\mathcal{D}(\Delta_{\alpha,P})$ acting via $u \mapsto \Delta u$. The extension $\Delta_{0,P}$ coincides with the Friedrichs extension and is nothing but the self-adjoint Laplacian on X_d .

Let $R(x, y; \lambda)$ be the resolvent kernel of the self-adjoint Laplacian Δ on X_d .

Following [3] define the scattering coefficient $F(\lambda; P)$ via

$$(2) \quad -R(x, P; \lambda) = G_d(r) + F(\lambda; P) + o(1)$$

as $x \rightarrow P$. (Notice that in [3] the resolvent is defined as $(\lambda - \Delta)^{-1}$, whereas for us it is $(\Delta - \lambda)^{-1}$. This results in the minus sign in (2).)

As it was already mentioned the deficiency indices of the symmetric operator Δ with domain $C_c^\infty(X_d \setminus \{P\})$ are $(1, 1)$, therefore, one has the following Krein formula (see, e. g., [1], p. 357) for the resolvent kernel, $R_\alpha(x, y; \lambda)$, of the self-adjoint extension $\Delta_{\alpha,P}$:

$$(3) \quad R_\alpha(x, y; \lambda) = R(x, y; \lambda) + k(\lambda; P)R(x, P; \lambda)R(P, y; \lambda)$$

with some $k(\lambda; P) \in \mathbb{C}$.

The following Lemma relates $k(\lambda; P)$ to the scattering coefficient $F(\lambda; P)$.

Lemma 1. *One has the relation*

$$(4) \quad k(\lambda; P) = \frac{\sin \alpha}{F(\lambda; P) \sin \alpha - \cos \alpha}.$$

Proof. Send $x \rightarrow P$ in (3), observing that $R_\alpha(\cdot, y; \lambda)$ belongs to $\mathcal{D}_{\alpha, P}$, make use of (1) and (2), and then compare the coefficients near $G_d(r)$ and the constant terms in the asymptotical expansions at the left and at the right. \square

It follows in particular from the Krein formula that the difference of the resolvents $(\Delta_{\alpha, P} - \lambda)^{-1} - (\Delta - \lambda)^{-1}$ is a rank one operator. The following simple Lemma is the key observation of the present work.

Lemma 2. *One has the relation*

$$(5) \quad \text{Tr} \left((\Delta_{\alpha, P} - \lambda)^{-1} - (\Delta - \lambda)^{-1} \right) = \frac{F'_\lambda(\lambda; P) \sin \alpha}{\cos \alpha - F(\lambda; P) \sin \alpha}.$$

Proof. One has

$$-F'_\lambda(\lambda; P) = \frac{\partial R(y, P; \lambda)}{\partial \lambda} \Big|_{y=P} = \lim_{\mu \rightarrow \lambda} \frac{R(y, P; \mu) - R(y, P; \lambda)}{\mu - \lambda}$$

Using resolvent identity we rewrite the last expression as

$$\lim_{\mu \rightarrow \lambda} \int_{X_d} R(y, z; \mu) R(P, z; \lambda) dz \Big|_{y=P} = \int_{X_d} [R(P, z; \lambda)]^2 dz$$

From (3) it follows that

$$[R(P, z; \lambda)]^2 = \frac{1}{k(\lambda; P)} (R_{\alpha, P}(x, z; \lambda) - R(x, z; \lambda)) \Big|_{x=z}.$$

This implies

$$-F'_\lambda(\lambda; P) = \frac{1}{k(\lambda, P)} \text{Tr} \left((\Delta_{\alpha, P} - \lambda)^{-1} - (\Delta - \lambda)^{-1} \right)$$

which, together with Lemma 1, imply (5). \square

Introduce the domain

$$\Omega_{\alpha, P} = \mathbb{C} \setminus \{ \lambda - it, \lambda \in \text{Spectrum}(\Delta) \cup \text{Spectrum}(\Delta_{\alpha, P}); t \in (-\infty, 0] \}.$$

Then in $\Omega_{\alpha, P}$ one can introduce the function

$$(6) \quad \tilde{\xi}(\lambda) = -\frac{1}{2\pi i} \log(\cos \alpha - F(\lambda; P) \sin \alpha)$$

(It should be noted that the function $\xi = \Re(\tilde{\xi})$ is the spectral shift function of Δ and $\Delta_{\alpha, P}$.) One can rewrite (5) as

$$(7) \quad \text{Tr} \left((\Delta_{\alpha, P} - \lambda)^{-1} - (\Delta - \lambda)^{-1} \right) = 2\pi i \tilde{\xi}'(\lambda)$$

3. OPERATOR ZETA-FUNCTION OF $\Delta_{\alpha, P}$

Denote by $\zeta(s, A)$ the zeta-function

$$\zeta(s, A) = \sum_{\mu_k \in \text{Spectrum}(A)} \frac{1}{\mu_k^s}$$

of the operator A . (We assume that the spectrum of A is discrete and does not contain 0.)

Take any $\tilde{\lambda}$ from $\mathbb{C} \setminus (\text{Spectrum}(\Delta_{\alpha,P}) \cup \text{Spectrum}(\Delta))$. From the results of [3] it follows that the function $\zeta(s, \Delta_{\alpha,P} - \tilde{\lambda})$ is defined for sufficiently large $\Re s$. It is well-known that $\zeta(s, \Delta - \tilde{\lambda})$ is meromorphic in \mathbb{C} .

The proof of the following lemma coincides verbatim with the proof of Proposition 5.9 from [7].

Lemma 3. *Suppose that the function $\tilde{\xi}'(\lambda)$ from (7) is $O(|\lambda|^{-1})$ as $\lambda \rightarrow -\infty$. Let $-C$ be a sufficiently large negative number and let $c_{\tilde{\lambda},\epsilon}$ be a contour encircling the cut $c_{\tilde{\lambda}}$ which starts from $-\infty + 0i$, follows the real line till $-C$ and then goes to $\tilde{\lambda}$ remaining in $\Omega_{\alpha,P}$. Assume that $\text{dist}(z, c_{\tilde{\lambda}}) = \epsilon$ for any $z \in c_{\tilde{\lambda},\epsilon}$. Let also*

$$\zeta_2(s) = \int_{c_{\tilde{\lambda},\epsilon;2}} (\lambda - \tilde{\lambda})^{-s} \tilde{\xi}'(\lambda) d\lambda,$$

where the integral at the right hand side is taken over the part $c_{\tilde{\lambda},\epsilon;2}$ of the contour $c_{\tilde{\lambda},\epsilon}$ lying in the half-plane $\{\lambda : \Re \lambda > -C\}$. Let

$$\hat{\zeta}_2(s) = \lim_{\epsilon \rightarrow 0} \zeta_2(s) = 2i \sin(\pi s) \int_{-C}^{\tilde{\lambda}} (\lambda - \tilde{\lambda})_0^{-s} \tilde{\xi}'(\lambda) d\lambda,$$

where $(\lambda - \tilde{\lambda})_0^{-s} = e^{-i\pi s} \lim_{\lambda \downarrow c_{\tilde{\lambda}}} (\lambda - \tilde{\lambda})^{-s}$. Then the function

(8)

$$R(s, \tilde{\lambda}) = \zeta(s, \Delta_{\alpha,P} - \tilde{\lambda}) - \zeta(s, \Delta - \tilde{\lambda}) - 2i \sin(\pi s) \int_{-\infty}^{-C} |\lambda|^{-s} \tilde{\xi}'(\lambda) d\lambda - \hat{\zeta}_2(s)$$

can be analytically continued to $\Re s > -1$ with $R(0, \tilde{\lambda}) = R'_s(0, \tilde{\lambda}) = 0$.

For completeness we give a sketch of proof. Using (7), one has for sufficiently large $\Re s$

$$\begin{aligned} \zeta(s, \Delta_{\alpha,P} - \tilde{\lambda}) - \zeta(s, \Delta - \tilde{\lambda}) &= \frac{1}{2\pi i} \int_{c_{\tilde{\lambda},\epsilon}} (\lambda - \tilde{\lambda})^{-s} \text{Tr}((\Delta_{\alpha,P} - \lambda)^{-1} - (\Delta - \lambda)^{-1}) d\lambda = \\ &= \int_{c_{\tilde{\lambda},\epsilon}} (\lambda - \tilde{\lambda})^{-s} \tilde{\xi}'(\lambda) d\lambda = \zeta_1(s) + \zeta_2(s), \end{aligned}$$

where

$$\zeta_1(s) = \left\{ \int_{-\infty+i\epsilon}^{-C+i\epsilon} - \int_{-\infty-i\epsilon}^{-C-i\epsilon} \right\} (\lambda - \tilde{\lambda})^{-s} \tilde{\xi}'(\lambda) d\lambda.$$

It is easy to show (see Lemma 5.8 in [7]) that in the limit $\epsilon \rightarrow 0$ $\zeta_1(s)$ gives

$$(9) \quad 2i \sin(\pi s) \int_{-\infty}^{-C} |\lambda|^{-s} \tilde{\xi}'(\lambda) d\lambda + 2i \sin(\pi s) \int_{-\infty}^{-C} |\lambda|^{-s} \tilde{\xi}'(\lambda) \rho(s, \tilde{\lambda}/\lambda) d\lambda,$$

where $\rho(s, z) = (1+z)^{-s} - 1$ and

$$\rho(s, \tilde{\lambda}/\lambda) = O(|\lambda|^{-1})$$

as $\lambda \rightarrow -\infty$. Using the assumption on the asymptotics of $\tilde{\xi}(\lambda)$ as $\lambda \rightarrow -\infty$ and the obvious relation $\rho(0, z) = 0$ one can see that the last term in (9) can be analytically continued to $\Re s > -1$ and vanishes together with its first derivative w. r. t. s at $s = 0$. Denoting it by $R(s, \tilde{\lambda})$ one gets the Lemma. \square

As it is stated in the introduction the main object we are to study in the present paper is the zeta-regularized determinant of the operator $\Delta_{\alpha, P} - \lambda$. Let us remind the reader that the usual definition of the zeta-regularized determinant of an operator A

$$(10) \quad \det A = \exp(-\zeta'(0, A))$$

requires analyticity of $\zeta(s, A)$ at $s = 0$.

Since the operator zeta-function $\zeta(s, \Delta - \tilde{\lambda})$ is regular at $s = 0$ (in fact, it is true in case of Δ being an arbitrary elliptic differential operator on any compact manifold) and the function $\hat{\zeta}_2(s)$ is entire, Lemma 3 shows that the behavior of the function $\zeta(s, \Delta_{\alpha, P} - \tilde{\lambda})$ at $s = 0$ is determined by the properties of the analytic continuation of the term

$$(11) \quad 2i \sin(\pi s) \int_{-\infty}^{-C} |\lambda|^{-s} \tilde{\xi}'(\lambda) d\lambda$$

in (8). These properties in their turn are determined by the asymptotical behavior of the function $\tilde{\xi}'(\lambda)$ as $\lambda \rightarrow -\infty$.

It turns out that the latter behavior depends on dimension d . In particular, in the next section we will find out that in case $d = 2$ the function $\zeta(s, \Delta_{\alpha, P} - \tilde{\lambda})$ is not regular at $s = 0$, therefore, in order to define $\det(\Delta_{\alpha, P} - \tilde{\lambda})$ one has to use a modified version of (10).

4. DETERMINANT OF PSEUDO-LAPLACIAN ON TWO-DIMENSIONAL COMPACT MANIFOLD

Let X be a two-dimensional Riemannian manifold, then introducing isothermal local coordinates (x, y) and setting $z = x + iy$, one can write the area element on X as

$$\rho^{-2}(z) |dz|^2$$

The following estimate of the resolvent kernel, $R(z', z; \lambda)$, of the Laplacian on X was found by J. Fay (see [5]; Theorem 2.7 on page 38 and the formula preceding Corollary 2.8 on page 39; notice that Fay works with negative Laplacian, so one has to take care of signs when using his formulas).

Lemma 4. (J. Fay) *The following equality holds true*

$$(12) \quad -R(z, z'; \lambda) = G_2(r) + \frac{1}{2\pi} \left[\gamma + \log \frac{\sqrt{|\lambda| + 1}}{2} \right. \\ \left. - \frac{1}{2(|\lambda| + 1)} \left(1 + \frac{4}{3} \rho^2(z) \partial_{z\bar{z}}^2 \rho(z) \right) + \hat{R}(z', z; \lambda) \right],$$

where $\hat{R}(z', z; \lambda)$ is continuous for z' near z ,

$$\hat{R}(z, z; \lambda) = O(|\lambda|^{-2})$$

uniformly w. r. t. $z \in X$ as $\lambda \rightarrow -\infty$; $r = \text{dist}(z, z')$, γ is the Euler constant.

Using (12), we immediately get the following asymptotics of the scattering coefficient $F(\lambda, P)$ as $\lambda \rightarrow -\infty$:

$$(13) \quad F(\lambda, P) = \frac{1}{4\pi} \log(|\lambda|+1) + \frac{\gamma - \log 2}{2\pi} - \frac{1}{4\pi(|\lambda|+1)} \left[1 + \frac{4}{3} \rho^2(z) \partial_{z\bar{z}}^2 \rho(z) \Big|_{z=z(P)} \right] + O(|\lambda|^{-2}).$$

Remark 1. It is easy to check that the expression $\rho^2(z) \partial_{z\bar{z}}^2 \rho(z) \Big|_{z=z(P)}$ is independent of the choice of conformal local parameter z near P .

Now from (6) and (13) it follows that

$$2\pi i \tilde{\xi}'(\lambda) = - \frac{\frac{1}{4\pi(|\lambda|+1)} - \frac{b}{(|\lambda|+1)^2} + O(|\lambda|^{-3})}{\cot \alpha - a - \frac{1}{4\pi} \log(|\lambda|+1) + \frac{b}{|\lambda|+1} + O(|\lambda|^{-2})},$$

where $a = \frac{1}{2\pi}(\gamma - \log 2)$ and $b = \frac{1}{4\pi}(1 + \frac{4}{3} \rho^2 \partial_{z\bar{z}}^2 \rho)$. This implies that for $-\infty < \lambda \leq -C$ one has

$$(14) \quad 2\pi i \tilde{\xi}'(\lambda) = \frac{1}{|\lambda|(\log |\lambda| - 4\pi \cot \alpha + 4\pi a)} + f(\lambda),$$

with $f(\lambda) = O(|\lambda|^{-2})$ as $\lambda \rightarrow -\infty$. Now knowing (14), one can study the behaviour of the term (11) in (8). We have

$$(15) \quad 2i \sin(\pi s) \int_{-\infty}^{-C} |\lambda|^{-s} \tilde{\xi}'(\lambda) d\lambda = \frac{\sin(\pi s)}{\pi} \int_{-\infty}^{-C} |\lambda|^{-s-1} \frac{d\lambda}{(\log |\lambda| - 4\pi \cot \alpha + 4\pi a)} + \frac{\sin(\pi s)}{\pi} \int_{-\infty}^{-C} |\lambda|^{-s} f(\lambda) d\lambda.$$

The first integral in the right hand side of (15) appeared in ([9], p. 15), where it was observed that it can be easily rewritten through the function

$$\text{Ei}(z) = - \int_{-z}^{\infty} e^{-y} \frac{dy}{y} = \gamma + \log(-z) + \sum_{k=1}^{\infty} \frac{z^k}{k \cdot k!}$$

which leads to the representation

$$(16) \quad \frac{\sin(\pi s)}{\pi} \int_{-\infty}^{-C} |\lambda|^{-s-1} \frac{d\lambda}{(\log |\lambda| - 4\pi \cot \alpha + 4\pi a)} = - \frac{\sin(\pi s)}{\pi} e^{-s\kappa} [\gamma + \log(s(\log C - \kappa)) + e(s)]$$

where $e(s)$ is an entire function such that $e(0) = 0$; $\kappa = 4\pi \cot \alpha - 4\pi a$. From this we conclude that

$$(17) \quad \frac{\sin(\pi s)}{\pi} \int_{-\infty}^{-C} |\lambda|^{-s-1} \frac{d\lambda}{(\log |\lambda| - 4\pi \cot \alpha + 4\pi a)} = -s \log s + g(s)$$

where $g(s)$ is differentiable at $s = 0$.

Now (8) and (17) justify the following definition.

Definition 1. Let $\Delta_{\alpha,P}$ be the pseudo-laplacian on a two-dimensional compact Riemannian manifold. Then the zeta-regularized determinant of the operator $\Delta_{\alpha,P} - \tilde{\lambda}$ with $\tilde{\lambda} \in \mathbb{C} \setminus \text{Spectrum}(\Delta_{\alpha,P})$ is defined as

$$(18) \quad \det(\Delta_{\alpha,P} - \tilde{\lambda}) = \exp \left\{ -\frac{d}{ds} \left[\zeta(s, \Delta_{\alpha,P} - \tilde{\lambda}) + s \log s \right] \Big|_{s=0} \right\}$$

We are ready to get our main result: the formula relating $\det(\Delta_{\alpha,P} - \tilde{\lambda})$ to $\det(\Delta - \tilde{\lambda})$.

From (8, 11) it follows that

$$(19) \quad \begin{aligned} & \frac{d}{ds} \left[\zeta(s, \Delta_{\alpha,P} - \tilde{\lambda}) + s \log s - \zeta(s, \Delta - \tilde{\lambda}) \right] \Big|_{s=0} = \\ & \frac{d}{ds} \hat{\zeta}_2(s) \Big|_{s=0} + \int_{-\infty}^{-C} f(\lambda) d\lambda + \\ & -\frac{d}{ds} \left\{ \frac{\sin \pi s}{\pi} e^{-s\kappa} [\gamma + \log(s(\log C - \kappa)) + e(s)] + s \log s \right\} \Big|_{s=0} = \\ & 2\pi i \left(\tilde{\xi}(\tilde{\lambda}) - \tilde{\xi}(-C) \right) + \int_{-\infty}^{-C} f(\lambda) d\lambda - \gamma - \log(\log C - \kappa) = \\ & 2\pi i \tilde{\xi}(\tilde{\lambda}) - \gamma + \\ & \int_{-\infty}^{-C} f(\lambda) d\lambda - 2\pi i \tilde{\xi}(-C) - \log(\log C - 4\pi \cot \alpha + 2\gamma - \log 4). \end{aligned}$$

Notice that the expression in the second line of (19) should not depend on C , so one can send C to $+\infty$ there. Together with (13) this gives

$$(20) \quad \begin{aligned} & \frac{d}{ds} \left[\zeta(s, \Delta_{\alpha,P} - \tilde{\lambda}) + s \log s - \zeta(s, \Delta - \tilde{\lambda}) \right] \Big|_{s=0} = \\ & 2\pi i \tilde{\xi}(\tilde{\lambda}) - \gamma + \log(\sin \alpha / (4\pi)) - i\pi \end{aligned}$$

which implies the comparison formula for the determinants stated in the following theorem.

Theorem 1. Let $\tilde{\lambda}$ do not belong to the union of spectra of Δ and $\Delta_{\alpha,P}$ and let the zeta-regularized determinant of $\Delta_{\alpha,P}$ be defined as in (18). Then one has the relation

$$(21) \quad \det(\Delta_{\alpha,P} - \tilde{\lambda}) = -4\pi e^{\gamma(\cot \alpha - F(\tilde{\lambda}, P))} \det(\Delta - \tilde{\lambda}).$$

Observe now that 0 is the simple eigenvalue of Δ and, therefore, it follows from Theorem 2 in [3] that 0 does not belong to the spectrum of the operator $\Delta_{\alpha,P}$ and that $\Delta_{\alpha,P}$ has one strictly negative simple eigenvalue. Thus, the determinant in the left hand side of (21) is well defined for $\tilde{\lambda} = 0$, whereas the determinant at the right hand side has the asymptotics

$$(22) \quad \det(\Delta - \tilde{\lambda}) \sim (-\tilde{\lambda}) \det^* \Delta$$

as $\tilde{\lambda} \rightarrow 0-$. Here $\det^* \Delta$ is the modified determinant of an operator with zero mode.

From the standard asymptotics

$$-R(x, y; \lambda) = \frac{1}{\text{Vol}(X)} \frac{1}{\lambda} + G_2(r) + O(1)$$

as $\lambda \rightarrow 0$ and $x \rightarrow y$ one gets the asymptotics

$$(23) \quad F(\lambda, P) = \frac{1}{\text{Vol}(X)} \frac{1}{\lambda} + O(1)$$

as $\lambda \rightarrow 0$. Now sending $\tilde{\lambda} \rightarrow 0-$ in (21) and using 22 and 23 we get the following corollary of the Theorem 1.

Corollary 1. *The following relation holds true*

$$(24) \quad \det \Delta_{\alpha,P} = -\frac{4\pi e^\gamma}{\text{Vol}(X)} \det^* \Delta.$$

5. DETERMINANT OF PSEUDO-LAPLACIAN ON THREE-DIMENSIONAL MANIFOLDS

Let X be a three-dimensional compact Riemannian manifold. We start with the Lemma describing the asymptotical behavior of the scattering coefficient as $\lambda \rightarrow -\infty$.

Lemma 5. *One has the asymptotics*

$$(25) \quad F(\lambda; P) = \frac{1}{4\pi} \sqrt{-\lambda} + c_1(P) \frac{1}{\sqrt{-\lambda}} + O(|\lambda|^{-1})$$

as $\lambda \rightarrow -\infty$

Proof. Consider Minakshisundaram-Pleijel asymptotic expansion ([12])

$$(26) \quad H(x, P; t) = (4\pi t)^{-3/2} e^{-d(x,P)^2/(4t)} \sum_{k=0}^{\infty} u_k(x, P) t^k$$

for the heat kernel in a small vicinity of P , here $d(x, P)$ is the geodesic distance from x to P , functions $u_k(\cdot, P)$ are smooth in a vicinity of P , the equality is understood in the sense of asymptotic expansions. We will make use of the standard relation

$$(27) \quad R(x, y; \lambda) = \int_0^{+\infty} H(x, y; t) e^{\lambda t} dt.$$

Let us first truncate the sum (26) at some fixed $k = N + 1$ so that the remainder, r_n , is $O(t^N)$. Defining

$$\tilde{R}_N(x, P; -\lambda) := \int_0^\infty r_n(t, x, P) e^{t\lambda} dt,$$

we see that

$$\tilde{R}_N(x, P; \lambda) = O(|\lambda|^{-(N+1)})$$

as $\lambda \rightarrow -\infty$ uniformly w. r. t. x belonging to a small vicinity of P .

Now, for each $0 \leq k \leq N + 1$ we have to address the following quantity

$$R_k(x, P; \lambda) := \frac{u_k(x, y)}{(4\pi)^{3/2}} \int_0^\infty t^{k-\frac{3}{2}} e^{-\frac{d(x, P)^2}{4t}} e^{\lambda t} dt.$$

According to identity (36) below one has

$$(28) \quad R_0(x, P; \lambda) = \frac{u_0(x, P)}{(4\pi)^{3/2}} \frac{2\sqrt{\pi}}{d(x, P)} e^{-d(x, P)\sqrt{-\lambda}} = \frac{1}{4\pi d(x, P)} - \frac{1}{4\pi} \sqrt{-\lambda} + o(1),$$

as $d(x, P) \rightarrow 0$. For $k \geq 1$ one has

$$(29) \quad R_k(x, P; \lambda) = \frac{u_k(x, P)}{(4\pi)^{3/2}} 2^{3/2-k} \left(\frac{d(x, P)}{\sqrt{-\lambda}} \right)^{k-1/2} K_{k-\frac{1}{2}}(d(x, P)\sqrt{-\lambda}) = -c_k(P) \frac{1}{(\sqrt{-\lambda})^{2k-1}} + o(1)$$

as $d(x, P) \rightarrow 0$ (see [2], p. 146, f-la 29). Now (25) follows from (27), (28) and (29). \square

Now from Lemma 5 it follows that

$$(30) \quad 2\pi i \tilde{\xi}'(\lambda) = -\frac{1}{2\lambda} + O(|\lambda|^{-3/2})$$

as $\lambda \rightarrow -\infty$, therefore, one can rewrite (11) as

$$(31) \quad \frac{\sin(\pi s)}{\pi} \left\{ \int_{-\infty}^{-C} |\lambda|^{-s} (2\pi i \tilde{\xi}'(\lambda) + \frac{1}{2\lambda}) d\lambda + \frac{C^{-s}}{2s} \right\}$$

which is obviously analytic in $\Re s > -\frac{1}{2}$. Thus, it follows from (8) that the function $\zeta(s, \Delta_{\alpha, P} - \tilde{\lambda})$ is regular at $s = 0$ and one can introduce the usual zeta-regularization

$$\det(\Delta_{\alpha, P} - \tilde{\lambda}) = \exp\{-\zeta'(0, \Delta_{\alpha, P} - \tilde{\lambda})\}$$

of $\det(\Delta_{\alpha, P} - \tilde{\lambda})$.

Moreover, differentiating (8) with respect to s at $s = 0$ similarly to (19) we get

$$\frac{d}{ds} \left[\zeta(s, \Delta_{\alpha, P} - \tilde{\lambda}) - \zeta(s, \Delta - \tilde{\lambda}) \right] \Big|_{s=0} =$$

$$2\pi i(\tilde{\xi}(\tilde{\lambda}) - \tilde{\xi}(-C)) + \int_{-\infty}^{-C} (2\pi i\tilde{\xi}'(\lambda) + \frac{1}{2\lambda})d\lambda - \frac{1}{2}\log C =$$

which reduces after sending $-C \rightarrow -\infty$ to

$$2\pi i\tilde{\xi}(\tilde{\lambda}) + \log \sin \alpha - \log(4\pi) + i\pi = -\log(\cot \alpha - F(\lambda; P)) - \log(4\pi) + i\pi$$

which implies the following theorem.

Theorem 2. *Let $\Delta_{\alpha, P}$ be the pseudo-laplacian on X and $\tilde{\lambda} \in \mathbb{C} \setminus (\text{Spectrum}(\Delta) \cup \text{Spectrum}(\Delta_{\alpha, P}))$. Then*

$$(32) \quad \det(\Delta_{\alpha, P} - \tilde{\lambda}) = -4\pi(\cot \alpha - F(\tilde{\lambda}; P))\det(\Delta - \tilde{\lambda}).$$

Sending $\tilde{\lambda} \rightarrow 0$ and noticing that relation (23) holds also in case $d = 3$ we get the following corollary.

Corollary 2.

$$(33) \quad \det \Delta_{\alpha, P} = -\frac{4\pi}{\text{Vol}(X)} \det^* \Delta.$$

In what follows we consider two examples of three-dimensional compact Riemannian manifolds for which there exist explicit expressions for the resolvent kernels: a flat torus and the round (unit) $3d$ -sphere. These manifolds are homogeneous, so, as it is shown in [3], the scattering coefficient $F(\lambda, P)$ is P -independent.

Example 1: Round $3d$ -sphere.

Lemma 6. *Let $X = S^3$ with usual round metric. Then there is the following explicit expression for scattering coefficient*

$$(34) \quad F(\lambda) = \frac{1}{4\pi} \coth\left(\pi\sqrt{-\lambda-1}\right) \cdot \sqrt{-\lambda-1}$$

and, therefore, one has the following asymptotics as $\lambda \rightarrow -\infty$

$$(35) \quad F(\lambda) = \frac{1}{4\pi} \sqrt{|\lambda|-1} + O(|\lambda|^{-\infty}).$$

Remark 2. The possibility of finding an explicit expression for $F(\lambda)$ for S^3 was mentioned in [3]. However we failed to find (34) in the literature.

Proof. We will make use the well-known identity (see, e. g., [2], p. 146, f-la 28):

$$(36) \quad \int_0^{+\infty} e^{\lambda t} t^{-3/2} e^{-\frac{d^2}{4t}} dt = 2 \frac{\sqrt{\pi}}{|d|} e^{-|d|\sqrt{-\lambda}},$$

for $\lambda < 0$ and $d \in \mathbb{R}$ and the following explicit formula for the operator kernel $e^{-t}H(x, y; t)$ of the operator $e^{-t(\Delta+1)}$, where Δ is the (positive) Laplacian on S^3 (see [4], (2.29)):

$$(37) \quad e^{-t}H(x, y; t) = -\frac{1}{2\pi} \frac{1}{\sin d(x, y)} \frac{\partial}{\partial z} \Big|_{z=d(x, y)} \Theta(z, t).$$

Here $d(x, y)$ is the geodesic distance between $x, y \in S^3$ and

$$\Theta(z, t) = \frac{1}{\sqrt{4\pi t}} \sum_{k=-\infty}^{+\infty} e^{-(z+2k\pi)^2/4t}$$

is the theta-function.

Denoting $d(x, y)$ by θ and using (37) and (36), one gets

$$\begin{aligned} R(x, y; \lambda - 1) &= \int_0^{+\infty} e^{\lambda t} e^{-t} H(x, y; t) dt = \\ &= \frac{1}{4\pi} \frac{1}{\sin \theta} \left(- \sum_{k < 0} e^{(\theta+2k\pi)\sqrt{-\lambda}} + \sum_{k \geq 0} e^{-(\theta+2k\pi)\sqrt{-\lambda}} \right) = \\ &= \frac{1}{4\pi} \frac{1}{\sin \theta} \frac{1}{1 - e^{-2\pi\sqrt{-\lambda}}} \left[-e^{-2\pi\sqrt{-\lambda}} e^{\theta\sqrt{-\lambda}} + e^{-\theta\sqrt{-\lambda}} \right] = \\ (38) \quad &= \frac{1}{4\pi\theta} - \frac{1}{4\pi} \frac{1 + e^{-2\pi\sqrt{-\lambda}}}{1 - e^{-2\pi\sqrt{-\lambda}}} \sqrt{-\lambda} + o(1) \end{aligned}$$

as $\theta \rightarrow 0$, which implies the Lemma. \square

Example 2: Flat 3d-tori. Let $\{\mathbf{A}, \mathbf{B}, \mathbf{C}\}$ be a basis of \mathbb{R}^3 and let T^3 be the quotient of \mathbb{R}^3 by the lattice $\{m\mathbf{A} + n\mathbf{B} + l\mathbf{C} : (m, n, l) \in \mathbb{Z}^3\}$ provided with the usual flat metric.

Notice that the free resolvent kernel in R^3 is

$$\frac{e^{-\sqrt{-\lambda}\|x-y\|}}{4\pi\|x-y\|}$$

and, therefore,

$$(39) \quad R(x, y; \lambda) = \frac{e^{-\sqrt{-\lambda}\|x-y\|}}{4\pi\|x-y\|} + \frac{1}{4\pi} \sum_{(m,n,l) \in \mathbb{Z}^3 \setminus (0,0,0)} \frac{e^{-\sqrt{-\lambda}\|x-y+m\mathbf{A}+n\mathbf{B}+l\mathbf{C}\|}}{\|x-y+m\mathbf{A}+n\mathbf{B}+l\mathbf{C}\|}.$$

From (39) it follows that

$$\begin{aligned} F(\lambda) &= \frac{1}{4\pi} \sqrt{-\lambda} - \frac{1}{4\pi} \sum_{(m,n,l) \in \mathbb{Z}^3 \setminus (0,0,0)} \frac{e^{-\sqrt{-\lambda}\|m\mathbf{A}+n\mathbf{B}+l\mathbf{C}\|}}{\|m\mathbf{A}+n\mathbf{B}+l\mathbf{C}\|} = \\ &= \frac{1}{4\pi} \sqrt{-\lambda} + O(|\lambda|^{-\infty}) \end{aligned}$$

as $\lambda \rightarrow -\infty$.

Remark 3. It should be noted that explicit expressions for $\det^* \Delta$ in case $X = S^3$ and $X = T^3$ are given in [13] and [6].

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